

# **Modeling Financial Scenarios: A Framework for the Actuarial Profession**

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## **ABSTRACT**

This paper summarizes the research project on *Modeling of Economic Series Coordinated with Interest Rate Scenarios* initiated by the joint request for proposals by the Casualty Actuarial Society and the Society of Actuaries. The project involved the construction of a financial scenario model that simulates a variety of economic variables over a 50 year period. The variables projected by this model include interest rates, inflation, equity returns, dividend yields, real estate returns, and unemployment rates. This paper contains a description of the key issues involved in modeling these series, a review of the primary literature in this area, an explanation of parameter selection issues, and an illustration of the model's output. The paper is intended to serve as a practical guide to understanding the financial scenario model in order to facilitate the use of this model for such actuarial applications as Dynamic Financial Analysis, development of solvency margins, cash flow testing, operational planning, and other financial analyses of insurer operations.

## **1. INTRODUCTION**

In May 2001, the Casualty Actuarial Society (CAS) and the Society of Actuaries (SOA) jointly issued a request for proposals on the research topic "Modeling of Economic Series Coordinated with Interest Rate Scenarios." There were several specific objectives of the request:

- review the previous literature in the area of economic scenario modeling;
- determine appropriate data sources and methodologies to enhance economic modeling efforts relevant to the actuarial profession; and,
- produce a working model of economic series, coordinated with interest rates, that could be made public and used by actuaries via the CAS / SOA websites to project future economic scenarios.

Categories of economic series to be modeled included interest rates, equity price levels, inflation rates, unemployment rates, and real estate price levels.

This topic is of considerable value to the actuarial profession given the interest in and substantial development of dynamic financial analysis (DFA). A key aspect of the DFA process is the ability to probabilistically express future economic and financial environments. By considering a variety of future economic conditions, actuaries can evaluate an insurer's

alternative operating decisions and their potential impact on corporate value. An important consideration in creating multiple scenarios is the recognition of the interdependencies between the various economic and financial series - for example, between equity returns and interest rate movements.

In the broader insurance community, a second benefit of this research is for regulatory and rating agency purposes, such as for use in cash flow testing. By testing across a wide range of potential scenarios, an insurer's cash position and liquidity can be evaluated over a variety of future alternative economic and financial environments.

Previous research has suggested the need for sophisticated tools to evaluate the financial condition of insurers. Santomero and Babbel (1997) review the financial risk management practices of both the life and property-liability insurers and finds that significant improvements are necessary. They find that even the most advanced insurers are not doing an effective job managing their financial risks. Research also shows that the potential consequences of the lack of risk measurement cannot be ignored. A study by the Casualty Actuarial Society Financial Analysis Committee (1989) discusses the potential impact of interest rate risk for property-liability insurers. Hodes and Feldblum (1996) also examine the effects of interest rate risk on the assets and liabilities of a property-liability insurer. Staking and Babbel (1995) find that significant work is needed to better understand the interest rate sensitivity of an insurer's surplus.

This paper provides a summary of the development of a scenario generation model, which is now available for public use. Full descriptions of the project, the research methodology, analytical implications, and the model itself – a spreadsheet-based stochastic simulation model – are available on the CAS website at: <http://casact.org/research/econ/>.

This paper is organized as follows. Section two discusses the key issues that were addressed during the model's development and reviews the literature in each of these important areas. Section three describes the underlying variables of the model, illustrates how each process is simulated, discusses how the default parameters of the process were selected, and provides sources of data for use in selecting the appropriate parameters. Section four briefly explains how to use the financial scenario model and discusses how to incorporate the model into other actuarial applications. Section five illustrates the use of the model, summarizes the output produced in one simulation, and includes a number of tabular and graphical displays of the output. Section six concludes the paper.

## 2. ISSUES AND LITERATURE REVIEW

There are many issues involved in building an integrated financial scenario model for actuarial use. This section reviews the literature in the modeling of the term structure and equity returns. In addition, the financial models in the actuarial literature are reviewed.

### *Term Structure Modeling*

Insurance companies have large investments in fixed income securities and their liabilities often have significant interest rate sensitivities. Therefore, any financial model of insurance operations must include an interest rate model at its core. This section describes some of the relevant research issues involved in term structure modeling.

The role of the financial scenario generator is not to explain past movements in interest rates, nor is the model attempting to perfectly predict interest rates in any future period in order to exploit potential trading profits.<sup>1</sup> Rather, the model purports to depict plausible interest rate scenarios which may be observed at some point in the future. Ideally, the model should allow for a wide variety of interest rate environments to which an insurer might be exposed.

The literature in the area of interest rate modeling is voluminous. One strand of the literature looks to explore the possibility of predictive power in the term structure. Fama (1984) uses forward rates in an attempt to forecast future spot rates. He finds evidence that very short-term (one-month) forward rates can forecast spot rates one month ahead. Fama and Bliss (1987) examine expected returns on U.S. Treasury securities with maturities of up to five years. They find that the one-year interest rate has a mean-reverting tendency, which results in one-year forward rates having some long-term forecasting power.

### *Historical Interest Rate Movements*

Other research reviews historical interest rate movements, in an attempt to determine general characteristics of plausible interest rate scenarios. Ahlgrim, D'Arcy and Gorvett (1999) review historical interest rate movements from 1953-1999, summarizing the key elements of these movements. Chapman and Pearson (2001) provide a similar review of history in an attempt to

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<sup>1</sup> It might be noted that trying to develop a model that mimics past rate movements may be a futile exercise since, despite the volume of research in the area, no tractable model has yet been shown to be satisfactory in accurately explaining history.

assess what is known about interest rate movements (or at least what is commonly accepted) and what is unknown (or unknowable). Litterman and Sheinkmann (1988) use principal component analysis to isolate the most important factors driving movements of the entire term structure.

Some of the findings of these studies include:

- Short-term interest rates are more volatile than long-term rates. Ahlgrim, D'Arcy, and Gorvett (1999) use statistics (such as standard deviation) to show that long-term rates tend to be somewhat tethered, while short-term rates tend to be much more dispersed. (A graphical presentation of historical interest rate movements is available at <http://www.business.uiuc.edu/~s-darcy/present/casdfa3/GraphShow.exe>).
- Interest rates appear to revert to some “average” level. For example, when interest rates are high, there is a tendency for rates to subsequently fall. Similarly, when rates are low, they later tend to increase. While economically plausible, Chapman and Pearson (2001) point out that due to a relatively short history of data, there is only weak support for mean reversion. If anything, evidence suggests that mean reversion is strong only in extreme interest rate environments (see also Chapman and Pearson (2000)).
- While interest rate movements are complex, 99% of the total variation in the term structure can be explained by three basic shifts. Litterman and Sheinkmann (1988) show that over 90% of the movement in the term structure can be explained by simple parallel shifts (called the *level* component). Adding a shift in the *slope* of the term structure improves explanatory power to over 95%. Finally, including U-shaped shifts (called *curvature*) explains over 99% of the variation observed in historical term structure movements. Chapman and Pearson (2001) confirm that these three factors are persistent over different time periods.
- Volatility of interest rates is related to the level of the short-term interest rate. Chapman and Pearson (2001) further point out that the appropriate measure for volatility depends on whether the period 1979-1982 -- when the Federal Reserve shifted policy from focusing on interest rates to controlling inflation, resulting in a rapid increase in interest rates -- is treated as an aberration or included in the sample period.

### *Equilibrium and Arbitrage Free Models*

Several popular models have been proposed to incorporate some of the characteristics of historical interest rate movements. Often these continuous time models are based on only one

stochastic factor, movements (changes) in the short-term interest rate (the instantaneous rate). A generic form of a one-factor term structure model is:

$$dr_t = \kappa(\theta - r_t)dt + \sigma r_t^\gamma dB_t \quad (2.1)$$

When  $\gamma = 0$ , this model is equivalent to the formulation of Vasicek (1977); when  $\gamma = 0.5$ , the model is the process proposed by Cox, Ingersoll, Ross (1985) (hereafter CIR). Equation (2.1) incorporates mean reversion. To see this, consider the case where the current level of the short-term rate ( $r_t$ ) is above the mean reversion level  $\theta$ . The change in the interest rate is then expected to be negative – interest rates are expected to fall. The speed of the reversion is determined by the parameter  $\kappa$ . If  $\gamma > 0$ , then interest rate volatility is related to the level of the interest rate. Chan, Karolyi, Longstaff, and Schwartz (1992) estimate this class of interest rate models and determine that based on monthly data from 1964-1989 the value of  $\gamma$  is approximately 1.5.

Models of the type shown in (2.1) are called “equilibrium models” since investors price bonds by responding to the known expectations of future interest rates. Using the assumed process for short-term rates, one can determine the yield on longer-term bonds by looking at the expected path of interest rates until the bond’s maturity. To determine the full term structure, one can price bonds of any maturity based on the expected evolution in short-term rates over the life of the bond<sup>2</sup>:

$$P(t, T) = E \left[ e^{-\int_t^T r_u du} \right] \quad (2.2)$$

where  $P(t, T)$  is the time  $t$  price of a bond with maturity  $(T - t)$ . One of the primary advantages of equilibrium models is that bond prices and many other interest rate contingent claims have closed-form analytic solutions. Vasicek and CIR evaluate equation (2.2) to find bond prices:

$$P(t, T) = A(t, T)e^{-r_t B(t, T)} \quad (2.3)$$

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<sup>2</sup> It should be noted that the expectations in (2.2) are evaluated under the risk neutral measure. See Chapter 9 of Tuckman (2002) for an introduction to risk neutral valuation of bonds.

where  $A(t,T)$  and  $B(t,T)$  are functions of the known process parameters  $\kappa$ ,  $\theta$ , and  $\sigma$ . Therefore, given a realized value for  $r_t$ , rates of all maturities can be obtained.

One immediate problem with equilibrium models of the term structure is that the resulting term structure is inconsistent with observed market prices, even if the parameters of the model are chosen carefully; while internally consistent, equilibrium models are at odds with the way the market is actually pricing bonds. Where equilibrium models generate the term structure as an output, “arbitrage free models” take the term structure as an input. All future interest rate paths are projected from the existing yield curve.

Ho and Lee (1986) discuss a discrete time model of the no arbitrage approach and include a time dependent drift so that observed market prices of all bonds can be replicated. The continuous time equivalent of the Ho-Lee model is:

$$dr_t = \theta(t)dt + \sigma dB_t \quad (2.4)$$

The time dependent drift ( $\theta(t)$ ) of the Ho and Lee model is selected so that expected future interest rates agree with market expectations as reflected in the existing term structure. This drift is closely related to implied forward rates. Hull and White (1990) use Ho and Lee’s (1986) time-dependent drift to extend the equilibrium models of Vasicek and CIR. The one-factor Hull-White model is:

$$dr_t = \kappa(\theta(t) - r_t)dt + \sigma dB_t \quad (2.5)$$

Heath, Jarrow, and Morton (1992) generalize the arbitrage free approach by allowing movements across the entire term structure rather than a single process for the short rate. HJM posit a family of forward rate processes,  $f(t,T)$ .

$$df(t,T) = \mu(t,T, f(t,T))dt + \sigma(t,T, f(t,T))dB_t \quad (2.6)$$

$$\text{where } f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \quad (2.7)$$

Choosing between an arbitrage-free term structure model and an equilibrium model often depends on the specific application. Despite their initial appeal, arbitrage free approaches often have disadvantages. Tuckman (1996) provides an excellent review of the advantages and disadvantages of equilibrium models vs. arbitrage free models. Some of these include:

- Arbitrage-free models are most useful for pricing purposes, especially interest rate derivatives. Since derivatives are priced against the underlying assets, a model that explicitly

captures the market prices of those underlying assets is superior to models that more or less ignore market values. Hull (2003) comments that equilibrium models are judged to be inferior since traders will have little confidence in the price of an option if the model cannot accurately price the underlying asset. Research supports this argument: Jegadeesh (1998) looks at the pricing of interest rate caps and determines that arbitrage-free models price interest rate caps more accurately than equilibrium models. Unfortunately, the pricing accuracy of arbitrage-free term structure models is based on short pricing horizons; there have been no formal comparative tests of the pricing accuracy using long-term assets.

- Fitton and McNatt (1997) comment that arbitrage-free models are most useful for short-term pricing applications when similar market data is readily available. Arbitrage-free models are intractable over long periods of time. With many arbitrage-free models, the forward rate plays a central role in the expected path of interest rates. Forward rates are related to the slope of the term structure and may exhibit strange behavior which significantly impacts projections of interest rate paths in arbitrage-free term structure models. For steeply sloped yield curves, the forward rate may become very large. For parts of the term structure that are downward sloping, the forward rate may even become negative. Especially for long-term projections, simulation paths may become extreme since the effects of small fluctuations in the term structure are magnified in long-term forward rates. For long-term analysis, equilibrium models are more appropriate.
- Arbitrage-free models also suffer from inconsistency across time (see Wilmott (1998) and Tuckman (1996)). As mentioned above, many arbitrage-free term structure models assume that the risk-free rate is closely related to the forward rate curve. If the model were correct, the forward rates would be the perfect predictors of future spot rates. On any projection date, the term structure implies different future spot rates, as well as volatilities of these rates. Clearly the actual path of interest rates will differ from the implied forward rate curve, which means that future projections make different assumptions about future spot rates and volatilities. Equilibrium models provide more consistent statements about interest rates over time.
- Determining the input into an arbitrage-free model is not straightforward. One usually considers the term structure implied by risk-free securities such as U.S. Treasuries. There are several difficulties in looking at U.S. Treasury data. First, market data gathered from STRIP

data are noisy, especially at long maturities. An alternative source for long-term interest rate data is to look at yields on long-term U.S. Treasury bonds. However, the liquidity of these long-term coupon bonds is suspect, and since on-the-run (the most current issue of a particular bond) Treasury securities typically have higher liquidity (and higher prices), yields of the longest maturity bonds are forced down. The forward rate curve initially reflects interest rate information for short-term rates, but for longer maturities, liquidity issues dominate. The result is a strangely shaped forward rate curve that can have significant undulations stemming from illiquidity. In addition, the future of 30-year bonds is uncertain, given the Treasury's termination of 30-year bond issues. Fewer points on the term structure make arbitrage-free models very sensitive to the market data and/or particularly vulnerable to market inefficiencies. Equilibrium models do not suffer from these "dirty" data issues.

- Depending on the specific arbitrage-free model, one may have to resort to numerical techniques such as simulation or interest rate trees to value contingent claims. Equilibrium models often have closed form solutions for common interest rate dependent securities.

### *Single- vs. Multi-factor Models*

The models presented above are all one-factor term structure models since there is only a single variable generating stochastic movements in interest rates. One problem with one-factor models is that the single source of uncertainty drives all term structure movements. As a result, yields of all maturities are perfectly correlated to the one stochastic factor and the range of potential yield curves is limited. The effects of multi-dimensional moves in the term structure can have serious consequences on a portfolio's value. Reitano (1992) demonstrates that even small non-parallel shifts in the yield curve can cause extreme changes in asset values.

Introducing additional sources of uncertainty (such as allowing the long end of the curve to fluctuate and/or introducing stochastic volatility) provide for a fuller range of yield curve movements and shapes. The downside is that introducing multiple dimensions of yield curve movements quickly increases the complexity and tractability of the model. Choosing the number of stochastic factors for a term structure model represents an important balance between accuracy and simplicity.

To illustrate an example of a multi-factor term structure model, Hull and White (1994) extend the one-factor Hull-White model (1990) to include a stochastic mean reversion level:

$$\begin{aligned} dr_t &= (\theta(t) + u_t - ar_t)dt + \sigma_1 dB_{1t} \\ du_t &= -bu_t dt + \sigma_2 dB_{2t} \end{aligned} \quad (2.8)$$

Similar to the one-factor Hull-White model, the instantaneous short term rate ( $r_t$ ) reverts to some reversion level ( $u_t$ ). However, the mean reversion level is also stochastic. If the mean reversion level is interpreted as an infinite maturity bond, the effect of introducing this second stochastic factor is to allow movements at both ends of the yield curve. Any correlation between short and long rates is accounted for in the correlation of the Brownian motion components of equation (2.8).

### *Summary of Term Structure Issues*

The final choice of term structure model is a decision which frequently elicits passionate debate. Decisions are needed to select among the various kinds of assumptions including matching the existing term structure (equilibrium vs. arbitrage-free model), the number of parameters employed, etc. In making these decisions, it is vital to bear in mind the application of the model. The choice of a term structure model is likely to be different for short-term applications which require precision and comparability to traded securities, as compared to long-term strategic planning exercises.

For this research, it is NOT intended that our model will be used for trading purposes. Rather, it is meant to give insurers a range of potential interest rate scenarios that are possible in the future. In selecting a term structure model for the financial scenario generator, we attempted to balance three important (and often opposing) goals: (1) mimicking the key historical characteristics of term structure movements, (2) generating the entire term structure for any future projection date, and (3) recognizing the desire for parsimony.

The first concern led us to a multifactor model which allows for some flexibility in yield curve shapes. While single factor models are often easier to describe and use, their restricted yield dynamics are too important for insurers to ignore. The second issue highlights the importance of interest rates of all time horizons, not of any specific key rates on the curve. Based on the realizations of the limited number of stochastic factors, we preferred term structure models that have closed form solutions for bond prices so that the entire term structure can be quickly and easily retrieved. When closed form solutions for bond yields are available, this allows users of the term structure model to track all interest rates on the yield curve during a

simulation, not a limited few. For example, users of a term structure model who are interested mortgage prepayment rates will be interested in the refinancing rate, which may be closely related to bond yields of specific maturities (such as 10 years). Other users may be concerned about crediting rates that are a function of historical 5-year interest rates. Without some explicit closed form solution, the modeler has no foundation to imply yields of different maturities from a limited set of stochastic factors. The two-factor, equilibrium model selected for the financial scenario model is described in the third section of this paper.

### **Equity Returns**

Similar to interest rates, there have been many studies that have looked at the behavior of equity returns. Shiller (2000) and Seigel (2002) analyze long-term patterns in stock returns and provide helpful analyses of long-term trends. Sornette (2003) examines the behavior of stock markets, investigating why complex systems, such as stock markets, crash.

Often, equity returns are assumed to follow a normal distribution. For example, in the development of their famous option pricing formula, Black and Scholes (1973) assume (continuously compounded) returns for stocks are normally distributed. However, historical observation of equity returns reveals that the distribution has “fatter tails” than predicted by the assumption of normality (Campbell, Lo, and MacKinlay (1997)).

A number of alternative assumptions have been proposed for stock movements. Alexander (2001) summarizes a variety of substitutes including GARCH processes and principal component analysis. Hardy (2001) uses a regime-switching model for stock returns and concludes that the performance of the regime-switching model is favorable relative to competing models. To motivate the rationale for Hardy’s (2001) model, consider the severe decline of the stock market in October 1987. This single observation may appear to be too “extreme” and very unlikely given a single variance assumption. Instead, suppose that equity returns at any point in time are generated from two distinct distributions, a “high volatility” regime or a “low volatility” regime. The chance of switching from one regime to the other over the next time step is dictated by transition probabilities. During times of economic instability, the returns on equities may be more uncertain, representing a transition to the high volatility regime. Thus, the observation from October 1987 may simply be a draw from the high volatility regime.

We use Hardy's (2001) approach for equity returns, but apply the regime switching process to excess returns – over and above the nominal risk free rate. At any point in time, the excess return of stocks is a draw from a normal distribution that is conditional on the current regime.<sup>3</sup> For each period, there is a matrix of probabilities that dictate the movement between regimes. While there is no limit to the number of regimes that can be embedded in the model, Hardy (2001) finds marginal improvement in fit when extending the equity return model to more than two regimes.

### Actuarial Models

Redington (1952) pioneered the work in modeling insurers. This early work introduced the concept of immunization against interest rate risk and introduced the “funnel of doubt” terminology to convey uncertainty in outcomes. Modern approaches of modeling (including this research) focus first on assumptions of the external economic and financial environment before incorporating the impact of these variables on the operations of the insurer.

Wilkie's (1986) model proposes inflation as the independent variable, using a first-order autoregressive model to simulate inflation. Wilkie (1986) links the realization of inflation with other variables using a cascade approach. Wilkie's original model (1986) includes (1) dividends, (2) dividend yields, and (3) interest rates.

Wilkie (1995) updates his earlier work by expanding on the structural form of the processes used to represent key variables in his “stochastic investment model.” The paper includes several appendices that fully develop the time series tools used throughout the presentation including cointegration, simultaneity, vector autoregression (VAR), autoregressive conditional heteroscedasticity (ARCH), and forecasting. Wilkie (1995) also estimates parameters for each equation of the model by looking at data from 1923-1994 and performs tests on competing models for fit. As in the 1986 model, Wilkie's updated model simulates inflation as an autoregressive process which drives all of the other economic variables including dividend yields, long-term interest rates, short-term interest rates, real estate returns, wages, and foreign exchange. One shortfall of the Wilkie model is the inconsistent relationships generated among inflation and short-term vs. long-term interest rates. In addition, the equity returns are based on

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<sup>3</sup> Ahlgrim and D'Arcy (2003) extend this regime switching approach to international equities.

an autoregressive process which leads to a distribution of returns that is much more compact than history indicates.

Hibbert, Mowbray, and Turnbull (2001) describe a model using modern financial technology that generates values for the term structure of interest rates (both real and nominal interest rates), inflation, equity returns, and dividend payouts. They use a two-factor model for both interest rates and inflation, a regime-switching model for equities, and a one-factor autoregressive dividend yield model. The paper discusses issues related to parameter selection and also illustrates a simulation under alternate parameters, comparing results with the Wilkie model.

Dynamic financial analysis (DFA) has become the label under which these financial models are combined with an insurer's operations when performing a variety of applications including pricing, reserve adequacy, and cash flow testing. D'Arcy, Gorvett, et. al. (1997, 1998) walks through the development of a public-access DFA model and illustrates the use of the model in a case study.

### 3. DESCRIPTIONS OF THE FINANCIAL SCENARIO GENERATOR AND DATA

In this section, detailed descriptions are provided for each of the economic time series included in our model. Embedded in these descriptions are references to the sources of historical time series data used to select the parameters of the model.

#### Inflation

Inflation (denoted by  $q$ ) is assumed to follow an Ornstein-Uhlenbeck process of the form (in continuous time):

$$dq_t = \kappa (\mu_q - q_t)dt + \sigma dB_q \quad (3.1)$$

The simulation model samples the discrete form equivalent of this process as:

$$\begin{aligned} \Delta q_t &= q_{t+1} - q_t = \kappa_q (\mu_q - q_t)\Delta t + \varepsilon_q \sigma_q \sqrt{\Delta t} \\ q_{t+1} &= q_t + \kappa_q (\mu_q - q_t)\Delta t + \varepsilon_q \sigma_q \sqrt{\Delta t} \\ &= \kappa_q \Delta t \cdot \mu_q + (1 - \kappa_q \Delta t) \cdot q_t + \varepsilon_q \sigma_q \sqrt{\Delta t} \end{aligned} \quad (3.2)$$

From this last equation, we can see that the expected level of future inflation is a weighted average between the most recent value of inflation ( $q_t$ ) and a mean reversion level of inflation,

$\mu_q$ . The speed of reversion is determined by the parameter  $\kappa_q$ . In the continuous model, mean reversion can be seen by considering the first term on the right-hand side of (3.1) (which is called the drift of the process). If the current level of inflation ( $q_t$ ) is above the mean reversion level, the first term is negative. Therefore, equation (3.1) predicts that the expected change in inflation will be negative; that is, inflation is expected to fall. The second term on the right-hand side of (3.1) represents the uncertainty in the process. The change in Brownian motion ( $dB_t$ ) can be likened to a draw from a standardized normal random variable (represented by  $\varepsilon_q$  in the discrete form of the model). The uncertainty is scaled by the parameter  $\sigma_q$ , which affects the magnitude of the volatility associated with the inflation process.

We can rearrange the last equation above to show that this process is an autoregressive process.

$$\begin{aligned} q_{t+1} - \mu_q &= \mu_q \kappa_q \Delta t - \mu_q + (1 - \kappa_q \Delta t) \cdot q_t + \varepsilon_q \sigma_q \sqrt{\Delta t} \\ &= (1 - \kappa_q \Delta t) \cdot q_t - (1 - \kappa_q \Delta t) \cdot \mu_q + \varepsilon_q \sigma_q \sqrt{\Delta t} \\ &= \phi_1 (q_t - \mu_q) + \varepsilon_q \sigma_q \sqrt{\Delta t} \end{aligned} \quad (3.3)$$

Using the last equation in (3.2), we can estimate the parameters of the inflation model using the following time series regression:

$$q_{t+1} = \alpha + \beta q_t + \varepsilon'_t \quad (3.4)$$

Note that we have not run the regression using the change in inflation as the dependent variable since this would not allow us to simultaneously derive the mean reversion speed ( $\kappa_q$ ) and the mean reversion level ( $\mu_q$ ). To derive the parameters of the inflation process, we transform the regression coefficients in (3.4):

$$\begin{aligned} \beta &= (1 - \kappa_q \Delta t) \\ \kappa_q &= \frac{1 - \beta}{\Delta t} \quad \text{and} \end{aligned} \quad (3.5)$$

$$\begin{aligned} \alpha &= \kappa_q \mu_q \Delta t = \frac{1 - \beta}{\Delta t} \mu_q \Delta t \\ \mu_q &= \frac{\alpha}{1 - \beta} \end{aligned} \quad (3.6)$$

We gathered inflation data from the Consumer Price Index (CPI) data collected by the Bureau of Labor statistics ([www.bls.gov](http://www.bls.gov)) and ran several regressions of this type to estimate  $\kappa_q$  and  $\mu_q$ .

One specific concern of this data was that individual *monthly* CPI levels might contain errors that would bias the regression coefficients. For example, if the CPI level of September 2004 was overstated, then inflation in September would appear “high” while the subsequent inference of inflation would appear “low”. If the time series of CPI contained any errors of this type, the resulting mean reversion strength and volatility parameters may be overstated. Given the noisy fluctuations in monthly data, we selected the parameters for the inflation process by looking at annual regressions. By calculating the change in CPI over the course of a year, the inflation rate would appear less volatile.

The often-cited time series of CPI uses a base period (i.e., resets the index value at 100) between the years 1982 and 1984. Given the fact that the CPI level is only reported to the first decimal place, using the current base does not lend itself to capturing minor changes in inflation in the first half of the 20<sup>th</sup>-century; a small change in CPI may lead to large swings in inflation when the level of the index is low. The only other publicly available series reported on the old base level (1967 = 100) is the one that is not seasonally adjusted, U.S. city averages, all items.

The annual rate of inflation was measured as:

$$q_t = \ln \frac{CPI_t}{CPI_{t-1}} \quad (3.7)$$

where  $CPI_t$  is the reported index value for year  $t$  and  $CPI_{t-1}$  is the prior year’s reported index value of the same month. We ran two annual regressions: (1) all available data and (2) the years after World War II.

<b><u>Time Period</u></b>	<b><math>\kappa_q</math></b>	<b><math>\mu_q</math></b>	<b><math>\sigma_q</math></b>
1913-2001	0.37	3.3%	4.0%
1946-2001	0.47	4.8%	3.0%

We selected the default mean reversion speed ( $\kappa_q$ ) to be 0.4 and the mean reversion level ( $\mu_q$ ) to be 4.8% to capture the post war economic period. Although it might appear that the speed of mean reversion over the second half of the 20th-century has increased, it should be noted that the standard error of the estimate of  $\kappa_q$  is higher than over the larger time period (which undoubtedly is due to fewer data points).

Instead of being concerned with the annualized, instantaneous level of inflation, bond investors are more concerned with the expected level of inflation over the life of their

investment. Given the existing level of inflation ( $q_t$ ) and the parameters of the assumed process in (3.1), we can derive expectations of future inflation over various horizons. Our process for inflation follows the same Ornstein-Uhlenbeck process as in Vasicek (1977), so we can develop a “term structure” of inflation analogous to equation (2.3). This term structure posits an expected inflation rate over various horizons. A term structure of inflation is needed to generate nominal interest rates, since investors are concerned not only about the time value of money, but also the erosion of purchasing power expected over the life of their investment.

### Real Interest rates

To derive real interest rates, we selected a simple case of the two-factor Hull-White model (equation (2.8)). In this model, the short-term rate (denoted by  $r$ ) reverts to a long-term rate (denoted by  $l$ ) that is itself stochastic. The long rate reverts to an average mean reversion level  $\mu_r$ .

$$\begin{aligned} dr_t &= \kappa_r(l_t - r_t)dt + \sigma_r dB_r \\ dl_t &= \kappa_l(\mu_r - l_t)dt + \sigma_l dB_l \end{aligned} \quad (3.8)$$

In order to estimate the parameters of the model, we look at the discrete analog of the model:

$$\begin{aligned} \Delta r_t &= \kappa_r(l_t - r_t)\Delta t + \sigma_r \varepsilon_{rt} \\ \Delta l_t &= \kappa_l(\mu_r - l_t)\Delta t + \sigma_l \varepsilon_{lt} \end{aligned} \quad (3.9)$$

$$\begin{aligned} r_{t+1} - r_t &= \kappa_r(l_t - r_t)\Delta t + \sigma_r \varepsilon_{rt} = (\kappa_r l_t + \kappa_r r_t)\Delta t + \sigma_r \varepsilon_{rt} \\ l_{t+1} - l_t &= \kappa_l(\mu_r - l_t)\Delta t + \sigma_l \varepsilon_{lt} = (\kappa_l \mu_r - \kappa_l l_t)\Delta t + \sigma_l \varepsilon_{lt} \end{aligned} \quad (3.10)$$

$$\begin{aligned} r_{t+1} &= r_t + (\kappa_r l_t + \kappa_r r_t)\Delta t + \sigma_r \varepsilon_{rt} \\ &= \kappa_r \Delta t \cdot l_t + (1 - \kappa_r \Delta t) \cdot r_t + \sigma_r \varepsilon_{rt} \\ l_{t+1} &= l_t + (\kappa_l \mu_r - \kappa_l l_t)\Delta t + \sigma_l \varepsilon_{lt} \\ &= \kappa_l \Delta t \cdot \mu_r + (1 - \kappa_l \Delta t) \cdot l_t + \sigma_l \varepsilon_{lt} \end{aligned} \quad (3.11)$$

From these equations, we can see that the short rate is again a weighted average between the current levels of  $r_t$  and the mean reversion factor  $l_t$ . The mean reversion factor is itself a weighted average of its long-term mean ( $\mu_r$ ) and its current value ( $l_t$ ).

Hibbert, Mowbray, and Turnbull (2001) (hereafter HMT) also use this process for real interest rates. They derive closed form solutions for bond prices (and therefore yields), which are slightly more complicated than the one-factor Ornstein-Uhlenbeck process for inflation:

$$P^r(t, T) = A^r(t, T)e^{-r_t B_1(t, T) - l_t B_2(t, T)} \quad (3.12)$$

where  $r_t$  and  $l_t$  are the values for the short and long real interest rate and  $A'$ ,  $B_1$ , and  $B_2$  are functions of underlying parameters in the two-factor Hull-White specification for real interest rates.

Estimating the equations in (3.11) is a difficult procedure since real interest rates are not directly observable in the market. We compute *ex post* real interest rates based on the difference between nominal rates observed in the market less the monthly (annualized) inflation rate. We use the three-month Constant Maturity Treasury (CMT) as a proxy for the instantaneous short rate and the 10-year CMT yield as a proxy for the long rate. (We also looked at longer Treasury yields as a proxy for the long rate. Results were not sensitive to the choice of maturity.) Nominal interest rates are from the Federal Reserve's historical database. (See <http://www.federalreserve.gov/releases/>).

There are several issues related to the Federal Reserve's interest rate data. First, at the long end of the yield curve, there are significant gaps in many of the time series. For example, the 20-year CMT was discontinued in 1987; yields on 20-year securities after 1987 would have to be interpolated from other yields. Also, the future of 30-year rate data is uncertain, given the decision of the Treasury to stop issuing 30-year bonds (in fact, the Fed's data stops reporting 30-year CMT data in the early 2002). At the short end of the yield curve, there are several choices for a proxy of the short rate. Ideally, one would want an interest-rate that most closely resembles a default-free instantaneous rate. While the one-month CMT is reported back only to 2001, the 3-month rate is available beginning in 1982. While we could have reverted to a private, proprietary source of data to create a longer time series, we restricted ourselves to only publicly available data sources that would be available to any user of the model.

We use the following regressions on monthly data from 1982 to 2001:

$$\begin{aligned} r_{t+1} &= \alpha_1 + \alpha_2 l_t + \alpha_3 r_t + \varepsilon'_{rt} \\ l_{t+1} &= \beta_1 + \beta_2 l_t + \varepsilon'_{lt} \end{aligned} \quad (3.13)$$

Traditional OLS regressions are not possible given the dependence of the short rate process on the long rate. To estimate these simultaneous equations, we use two-stage least squares estimation. In order to estimate the short-rate equation in stage 2, we first obtain estimates for the long-rate  $\hat{l}_t$ , based on the parameter estimates from stage 1.

$$\begin{aligned} \text{Stage 1: } l_{t+1} &= \beta_1 + \beta_2 l_t + \varepsilon'_{2t} \\ \text{Stage 2: } \Delta r_{t+1} &= \alpha_1 (\hat{l}_t - r_t) + \varepsilon'_{1t} \end{aligned} \quad (3.14)$$

The resulting parameters were generated from the regression results.

***Real Interest Rate Process  
Estimated from 1982 - 2001***

$\kappa_r$	$\mu_r$	$\sigma_r$	$\kappa_l$	$\sigma_l$
6.1	2.8%	10.0%	5.1	10.0%

These parameters indicate a very high level of volatility that is tempered by strong levels of mean reversion. See the discussion of the nominal interest rates below for the parameters that are used in the simulation illustration in section five.

**Nominal interest rates**

Fisher (1930) provides a thorough presentation of the interaction of real interest rates and inflation and their effects on nominal interest rates. He argues that nominal interest rates compensate investors not only for the time value of money, but also for the erosion of purchasing power that results from inflation. In the model presented here, the underlying movements in inflation and real interest rates generate the process for nominal interest rates. If bonds are priced using expectations of inflation and real interest rates until the bond's maturity, then nominal interest rates are implied by combining the term structure of inflation and the term structure of real interest rates. Therefore:

$$P^i(t, T) = P^r(t, T) \times P^q(t, T) \tag{3.15}$$

where  $i$  refers to nominal interest rates and the superscripts on the bond prices correspond to the underlying stochastic variables.

Unfortunately, the parameters for the real interest rate process shown above generate a distribution that severely restricts the range of potential future nominal interest rates. For example, using the regression results from equations (3.13) and (3.14), the 1<sup>st</sup> percentile of the distribution for the 20-year nominal rate is 5.9% and the 99<sup>th</sup> percentile is 8.2%. There are several candidates for problems with real interest rates that may lead to this seemingly unrealistic distribution of future nominal rates: (1) the use of *ex post* real interest rate measures is unsuitable, (2) because of potential errors in monthly reporting of CPI mentioned above, monthly

measurement of real interest rates produce self-correcting errors which exaggerate mean reversion speed, or (3) the time period used to measure real interest rates is too short.

As a result, the parameters for real interest rates were altered to allow nominal interest rates to better reflect historical volatility. Specifically, mean reversion speed was dramatically reduced. Given that mean reversion speed and volatility work together to affect the range of interest rate projections, volatility was also reduced. The following parameters are used as the “base case” in the model. These parameters are in line with what was used in Hull (2003).

$\kappa_r$	$\mu_r$	$\sigma_r$	$\kappa_i$	$\sigma_i$
1.0	2.8%	1.00%	0.1	1.65%

An important consideration in the model is the correlation between interest rates and inflation. Risa (2001) reviews the literature on the relationship between inflation and interest rates. Pennacchi (1991) finds evidence that the instantaneous real interest rates and expected inflation are significantly negatively correlated. Fama (1990) examines how one-year spot interest rates can be used to forecast its components: the one-year inflation rate and the real return on one-year bonds. It is found that the expected values of those two components move opposite to one another. As a result, the financial scenario model includes a negative correlation between real interest rates and inflation.

### Equity Returns

Equity returns are equal to the risk-free nominal interest rate ( $q + r$ ) and a risk premium or excess equity return attributable to capital appreciation ( $x$ ):

$$s_t = q_t + r_t + x_t \quad (3.16)$$

In her model, Hardy (2001) assumes that stock prices are lognormally distributed under each regime. But while Hardy looks at total equity returns, including dividends and the underlying compensation from the risk free rate, we use the excess equity returns  $x$ . To estimate the parameters of the regime switching equity return model, we follow the procedure outlined in Hardy (2001), maximizing the likelihood function implied from the regime switching process.

We estimate the process for the returns of small stocks and large stocks separately. Numerous web sites are available to capture the time series of capital appreciation of these

indices (see for example, finance.yahoo.com). We used the longest time series available for large stocks (1871-2002), available at Robert Shiller's web site. (See [http://www.econ.yale.edu/~shiller/data/ie\\_data.htm](http://www.econ.yale.edu/~shiller/data/ie_data.htm)). To look at small stocks, we used Ibbotson data captured from 1926-1999. As expected, the risk and return of small stocks appears higher than large stocks under both regimes. The following parameter estimates were developed:

### **Excess Monthly Returns**

	<i>Large Stocks (1871-2002)</i>		<i>Small Stocks (1926-1999)</i>	
	<i>Low Volatility Regime</i>	<i>High Volatility Regime</i>	<i>Low Volatility Regime</i>	<i>High Volatility Regime</i>
<i>Mean</i>	0.8%	-1.1%	1.0%	0.3%
<i>Standard Deviation</i>	3.9%	11.3%	5.2%	16.6%
<i>Probability of Switching</i>	1.1%	5.9%	2.4%	10.0%

Note that while the expected return in the high volatility regime is lower, it is more likely that if the high volatility regime is ever reached, the equity market will revert back to the low volatility regime since the probability of switching is higher. The regimes switches are correlated, so if large stocks are in the low volatility regime, then small stocks are more likely to be in the low volatility regime as well.

### **Equity Dividend Yields**

Similar to the process used by HMT and Wilkie (1986), we assume that (the log of) the dividend yield follows an autoregressive process.

$$d(\ln y_t) = \kappa_y (\mu_y - \ln y_t) dt + \sigma_y dB_{y_t} \quad (3.17)$$

One source of difficulty associated with estimating the dividend yield process involves obtaining data. There is no long time series of dividend yields that is publicly available for equity indices. To obtain this information, we used a proprietary source of financial data (Telerate). However, one may be able to estimate the dividend yield of indices that contained a limited number of stocks (such as the Dow Jones industrial average). It should be noted that the process for dividend yields is clearly time-dependent. Average dividend yields have fallen dramatically over

the last 50 years given the recognition of double taxation effects. Recent tax changes that levy lower taxes on dividends may (or may not) reverse the long-term trends of lower dividends.

Estimation of this process is analogous to the inflation process described above. The mean reversion speed of the series is not significantly different from zero. Given the long-term changes in historical dividend patterns, the (log of) dividends appear to be a random walk around its starting value.

### **Real Estate (Property)**

Given the that the real estate portfolios of insurers are dominated by commercial properties, we use the National Council of Real Estate Investment Fiduciaries (NCREIF) pricing index to capture the quarterly returns on commercial properties (see [www.ncreif.com](http://www.ncreif.com)). The NCREIF data is generated from market appraisals of various property types including apartment, industrial, office, and retail. While the use of appraisal data may only approximate sharp fluctuations in market valuation, publicly obtainable transaction-based real estate data were not available.

Using *quarterly* return data from NCREIF from 1978 to 2001, we estimated two separate Ornstein-Uhlenbeck models for real estate: the first model included the level of inflation while the second model did not.

$$d(re)_t = \kappa_{re} (\mu_{re} - (re)_t) dt + (\alpha q_t + ) \sigma_{re} dB_{re} \quad (3.18)$$

While we expected inflation to provide additional explanatory power for real estate returns, the results were not significant.

### **Unemployment**

There are many plausible ways to link unemployment rates to other economic variables. One approach to estimating unemployment is based on the well-known Phillips curve. The Phillips curve illustrates a common inverse relationship between unemployment and inflation. The approach taken by Phillips seems plausible: As the economy picks up, inflation increases to help temper the demand driven economy. At the same time, unemployment falls as firms hire to meet the increasing demand. When the economy slows down, unemployment rises, and inflationary pressures subside.

We include a first-order autoregressive process in the Phillips curve:

$$du_t = \kappa_u (\mu_u - u_t)dt + \alpha_u dq_t + \sigma_u \varepsilon_{ut} \quad (3.19)$$

It is expected that when inflation increases ( $dq_t > 0$ ), unemployment decreases (i.e.,  $\alpha_u < 0$ ). One may argue that there is a lag between inflation and unemployment. To keep the model simple, we did not pursue any distributed lag approach.

The discrete form of the unemployment model:

$$\begin{aligned} u_{t+1} &= u_t + \kappa_u \mu_u - \kappa_u \Delta t \cdot u_t + \alpha_u (q_{t+1} - q_t) + \sigma_u \varepsilon_{ut} \\ &= \kappa_u \mu_u + (1 - \kappa_u \Delta t) \cdot u_t + \alpha_u (q_{t+1} - q_t) + \sigma_u \varepsilon_{ut} \end{aligned} \quad (3.20)$$

This suggests the following regression:

$$u_{t+1} = \beta_1 + \beta_2 u_t + \beta_3 (q_{t+1} - q_t) + \sigma_u \varepsilon_{ut} \quad (3.21)$$

We use inflation data as described above and retrieve monthly unemployment data from the Bureau of Labor Statistics ([www.bls.gov](http://www.bls.gov)). Using data from 1948 to 2001 and transforming the regression coefficients as in (2.3), we get:

$$du_t = 0.13 \times (6.1\% - u_t)dt - 0.72dq_t + 0.76\% \times dB_{ut} \quad (3.22)$$

### **Comments on Selecting Parameters of the Model**

Some have argued that the performance of any model should be measured by comparing projected results against history. It is not our attempt to perfectly match the distribution of historical values for interest rates, equity returns, etc. To do so would naively predict a future based on random draws from the past. If perfect fit is desired, history already provides the set of economic scenarios that may be used for actuarial applications and the development of an integrated financial scenario model is completely unnecessary. Instead, the model presented here provides an alternative: an integrated approach to creating alternative scenarios which are tractable and realistic. While history is used to gain important insights into the characteristics of relevant variables, it would be impossible to build tractable models that yield a perfect fit to historical distributions. In general, we believe our theoretical framework provides a parsimonious approach to closed-form solutions of particular variables of interest.

## **4. USING THE FINANCIAL SCENARIO MODEL**

The financial scenario model is an Excel spreadsheet that benefits from the use of a simulation software package called @RISK available through Palisade Corporation

(www.palisade.com). @RISK leverages the simplicity of spreadsheets and integrates powerful analysis tools that are used to help randomly select future scenarios and examine risk in a stochastic financial environment. @RISK allows users to define uncertain variables as a distribution, take numerous draws from these inputs, and then capture each iteration's impact on a user defined output variable of interest, such as profits, sales, or an insurer's surplus.

### **Excluding Negative Nominal Interest Rates**

There has been significant debate over the proper way to deal with negative nominal interest rates in interest rate models. Some modelers have set boundary conditions that prevent nominal interest rates from becoming negative. Other modelers have not been concerned over negative interest rates, either because the mathematical characteristics of the model are more important than the practical applications, or the incidence of negative nominal interest rates is too infrequent to require significant attention.

While it depends on the specific application, the occurrence of negative nominal interest rates can be problematical. Economically, certain variables have natural limits. For example, while theory may not reject negative interest rates, reality suggests that it is unlikely that investors would ever accept negative nominal interest rates when lending money. Therefore, the model provides users with two options:

- *Placing lower bounds on the levels of inflation and real interest rates.* The model simulates these processes as if there were no lower bound, but then chooses the maximum of the lower bound and the simulated value.
- *Eliminating the potential for negative nominal interest rates.* In this case, the model uses the standard inflation simulation, but effectively places a lower bound on the real interest rates such that the resulting nominal interest rate is non-negative.

### **User Defined Scenarios**

The financial scenario model provides for stochastic simulation of future economic variables, based upon user-specified parameters for the assumed processes. However, there are instances where it may be desirable to allow the user to input specific scenarios for the future values of certain processes. For example, regulations may require sensitivity testing based on specific equity return patterns over the next decade. The financial scenario model allows users to

specify scenarios for three economic variables in the model: nominal interest rates, inflation, and equity returns. For example, with respect to nominal interest rates, each of the “New York 7” regulatory interest rate tests are pre-programmed into the model and may be selected by the user; the user may also specify a scenario of her/his own creation for any of the three economic processes.

### **Employing the Financial Scenario Model**

It is expected that the financial scenario model will be implemented in a variety of different analyses. The model can be used as the underlying engine for creating many financial scenarios and can be tailored for a user’s specific purposes. For example, Ahlgrim and D’Arcy (2003) use the model as the underlying asset return generator to assess the risk inherent in pension obligation bonds issued by the State of Illinois. In this case, the model was extended to include international equities and to compute yields on coupon bonds from the nominal interest rates.

## **5. ILLUSTRATIVE SIMULATION RESULTS**

Regardless of the mathematical sophistication of the variables incorporated in a model, the accuracy of the procedures used to determine the parameters, and the timeliness of the values on which the calibration is based, the most important test of the validity of any model is the reasonability of the results. This section will examine the results of a representative run of the financial scenario model and compare the output with historical values. It should be reiterated that the goal of choosing the parameters for the model was *not* to replicate history.

Correspondingly, we do not include measures of fit when comparing the sample results to history. This section uses history to review results of an illustrative simulation to subjectively assess the model’s plausibility.

A simulation is performed generating 5,000 iterations (sample paths) using the base parameters described in Section 3, disallowing negative nominal interest rates.<sup>4</sup> The results are presented in several different ways (these results are discussed in the following section).

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<sup>4</sup> The output of this illustration has been saved in a file and is posted at <http://casact.org/research/econ>. The American Academy of Actuaries use a similar prepackaged scenario approach in looking at C-3 risk of life insurers.

- Table 1 provides key statistics for key variables in the simulation. Mean values of the output are shown for the first and last (50<sup>th</sup>) projection years. The 1<sup>st</sup> and 99<sup>th</sup> percentiles of the distribution of results are indicated for an intermediate projection year (year 10).
- Tables 2 and 3 show the correlation matrices, comparing the simulation values (Table 2) and historical correlations (Table 3).
- Some of the Figures (1-6, 8-10, 14-15, 18, 20, and 22) show “funnel of doubt” plots, indicating the level of uncertainty surrounding each output variable *over time*.<sup>5</sup> The x-axis indicates the time period and the y-axis indicates the value(s) assumed by the variable of interest. The “funnel of doubt” graphs show the mean value for the 5,000 iterations (solid line) the 25<sup>th</sup> and 75<sup>th</sup> percentile values (dark shaded section) and the 1<sup>st</sup> and 99<sup>th</sup> percentile values (lighter shaded section). Expanding funnels indicate that the values become more uncertain over the projection period. Narrowing funnels indicate that the variables become more predictable when making long-term forecasts.
- Figures 7, 11-13, 16-17, 19, and 21 are histograms, illustrating the full probability distribution of the values for a particular variable *at one point in time* (a single projection year). For comparative purpose, the distribution of historical values, where appropriate, is also plotted in these histograms.

### **Real Interest Rates**

We start by looking at the 1-month real interest rate. The mean value for the first projection month is 0%. By the end of the 50 year projection period, this value has moved to 3.0%. This result is entirely in line with the specifications of the model. The one month value would be closely aligned with the initial short-term real interest rate (*rinit1*). To estimate this rate, we backed out an estimate of inflation from the observed risk-free, short-term interest rate. During the summer of 2004, the resulting value of the real interest rate was near 0%. Under the projections, the initial value would begin to revert to the long-term mean after one month. The mean of the final value in the results, after 50 years, is around the mean reversion level for the long rate (*rm2*), which is 2.8%.

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<sup>5</sup> These “funnel of doubt” graphs are referred to as “summary graphs” in @RISK.

To provide an idea about the range of values for the 1-month real interest rate, columns 3 and 4 of Table 1 display the 1<sup>st</sup> and 99<sup>th</sup> percentiles of the distribution in the tenth projection year. In 1 percent of the iterations, the 1-month real interest rate, on an annualized basis, is -5.3%. On first observation, this result seems nonsensical. Why would an investor be willing to lose money, in real terms, by investing at a negative real interest rate? Instead, an investor would just hold cash rather than lose 5.3% per year, after adjusting for inflation. However, this may not be as unrealistic as it seems. First, this result is annualized rate vs. the one-month real rate of only -0.4%. Second, this return may represent the best return available. If inflation is high, then holding cash would generate an even larger loss. In times of high inflation, the best real return an investor can receive may be negative. Finally, real interest rates are not observable. The true real interest rate is the return required, over and above expected inflation, for the specific interval. However, the precise expected inflation rate is unobservable in the financial markets.

In practice, two approaches have been used for estimating the expected inflation rate. First, one can use economists' forecasts of inflation. Economists, though, do not represent investors. By training and occupation, the economists included in the surveys are not at all representative of the general financial market participants. Investors may consider some economists' forecasts in making their own determination of what to expect regarding future economic conditions, but many other factors, including their own experience, the counsel of other participants, and recent historical experience, are used to determine their inflation expectations. There is no survey of representative market participants to determine what they truly anticipate for the inflation rate.

The second approach has been to examine actual inflation rates that have occurred, and then subtract those from prior interest rates (*ex post* analysis). This approach is also flawed for several reasons. First, there is no reason to believe that the market is prescient regarding inflation expectations. Especially in the case of an unexpected shock to the system, such as oil price increases during the 1970s, the market does not know what will happen in the future. It cannot even be assumed that errors in forecasting will cancel out over time, since the market could be biased to under, or over, estimate future inflation. Second, actual inflation cannot be accurately measured. The Consumer Price Index and other values commonly used to determine inflation are widely recognized as being imperfect. These indices track the prices of specific goods and services that are not completely representative of the entire economy. These indices

cannot recognize the substitution effect in which consumers continually engage, such as buying more chicken than usual when beef prices rise, or driving less when gasoline prices soar. Due to these problems, it is not possible to claim that real interest rates cannot be negative, so a small negative value over a short time interval does not necessarily represent a problem.

On the opposite side of the distribution, the 99<sup>th</sup> percentile value for 1-month real interest rates after 10 years is 10%. The same limitations described above also apply to this value.

Going further out on the term structure, the mean value of the 1-year real interest rate in the first projection is 0.3%. This reflects reversion from the initial value of 0% to the long-term mean of 2.8%. The mean of the last value, after 50 years, is, in line with these parameters, 2.9%. The 1<sup>st</sup> – 99<sup>th</sup> percentile range after 10 years is -5.1% to +9.7%, reflecting a similar distribution for the full year as was observed for the monthly values. For the 10-year real interest rates, the mean after the first projection month is 1.1% and the in the last projection month, the mean is 2.6%, reflecting the strength of the mean reversion over this long a period of time. The 1<sup>st</sup> – 99<sup>th</sup> percentile range after 10 years is -3.3% to +7.6%, reflecting the more compact distribution for long-term (10 year) real interest rates, compared to shorter time horizons.

Figures 1-3 depict the funnel of doubt graphs of 1-month, 1-year and 10-year real interest rates. All reflect the same shape, although the scaling differs. The “kink” in the early portion of the graph occurs because the first 12 points represent monthly intervals, which have small changes in values, and the latter steps are larger intervals, which lead to correspondingly larger changes. The level of uncertainty increases over the entire 50 year time frame, but the shifts toward the end of the simulation period are less pronounced. This shape occurs due to the structure and parameterization of the model. The uncertainty inherent in the real interest rate process generates the initial spread of the distribution, but the impact of mean reversion offsets this tendency, keeping the “funnel of doubt” from expanding further.

### **Inflation**

The next variable of interest is the inflation rate. As shown in Table 1, the mean value of the (annualized) 1-month inflation rate is 1.1% after only one month and 4.8% after 50 years. Note that the initial inflation rate (qinit1) is set at 1.0% and 4.8% is the long-term mean (qm2). The 1<sup>st</sup> – 99<sup>th</sup> percentile range after 10 years is -5.3% to +14.5%, which is wider than the distribution for real interest rates since the mean reversion speed for inflation is lower (0.4

compared to 1.0). Negative inflation (or deflation) is not objectionable since small negative monthly values have occurred in recent years. Monthly inflation values in excess of 14.5% did occur during the 1970s.

The mean 1-year inflation rate begins at 1.6% and moves to 4.8% by the end of 50 years, again both in line with the model parameters. The 1<sup>st</sup> – 99<sup>th</sup> percentile range of the 1-year inflation rate after 10 years is –3.7% to +12.9%. Although the United States has not experienced deflation over an entire year since 1954, it seems quite appropriate to assign positive probability to this event.

From the description in Section 3, recall that the 10-year inflation rate is derived from the expected path of inflation over the next ten years. Given the assumption of mean reversion of inflation, it is expected that there is less uncertainty inherent in predicting longer term inflation rates. The simulation confirms this – the mean 10-year inflation rate begins at 3.6% and moves to 4.5% by the end of 50 years, closer to the long-term mean parameter of 4.8%. Also, the 1<sup>st</sup> – 99<sup>th</sup> percentile range of the 10-year inflation rate after 10 years is 2.0% to 6.9%, demonstrating that, over longer time horizons, the (geometric) average rate of inflation is less variable.

The funnel of doubt graphs of 1-month, 1-year, and 10-year inflation rates are shown on Figures 4-6. The uncertainty of the 10-year inflation rate is much smaller than it is for 1-month and 1-year rates, reflecting the strength of the mean reversion term for this single factor model. Although inflation varies widely over shorter time horizons, in this model the long-term inflation rate is much less variable. This pattern can be altered by increasing the volatility of the inflation process ( $\sigma_q$ ) or reducing the mean reversion speed ( $\kappa_q$ ).

The histograms for the 1-year projected inflation rates and of actual 1-year inflation rates from 1913-2003 (from January to January) are shown on Figure 7. It is readily apparent that the modeled inflation rates generate a nice bell-shaped curve, whereas the actual inflation rates are much less smooth. One reason for this difference is that the model results are based on 5,000 iterations, while the actual data contain only 90 data points. More importantly, though, the projected values are derived from a concise mathematical expression that will produce a smooth distribution of results, but the actual inflation rates depend on the interactions of an almost unlimited number of variables. The key question, though, is whether the model adequately expresses the probability distribution of potential inflation rates. The actual inflation rates are

more leptokurtic (fatter in the tails than a normal distribution) than the modeled values, but reflect the central portion of the graph fairly well. All of the large negative inflation rates occurred prior to 1950. Many of the positive outliers are from years prior to 1980, when monetary policy was less focused on controlling inflation.

### **Nominal Interest Rates**

Nominal interest rates reflect the combination of the real interest rate and inflation. The mean values for 1-month nominal interest rates were 1.1% for the first month and 7.8% for the 50<sup>th</sup> year. The initial nominal interest rate indicated in the model (1.1%) is in line with the user defined starting level (June 2004) of 1.1%. The 1<sup>st</sup> – 99<sup>th</sup> percentile range for the 1-month nominal interest rate after 10 years is 0% to 19.4%.

The mean 1-year nominal interest begins at 1.9% and moves to 7.7% by the end of 50 years. The initial value is again in line with the current level of interest rates. The 1<sup>st</sup> – 99<sup>th</sup> percentile range of the 1-year nominal interest rate after 10 years is 0% to 18.3%.

The mean 10-year nominal interest begins at 4.6% and moves to 7.1% by the end of 50 years. The initial value is in line with the current level of interest rates for long-term bonds, given the June 2004 10-year U. S. Treasury yield of 4.4%. The 1<sup>st</sup> – 99<sup>th</sup> percentile range of the 1-year nominal interest rate after 10 years is 0.6% to 12.7%.

The Funnel of Doubt graphs of 1-month, 1-year, and 10-year nominal interest rates, Figures 8-10, are similar to the real interest rate and inflation graphs, but have a barrier at zero since the restriction that nominal interest rates not be negative is applied in this case. This restriction is illustrated by the 1<sup>st</sup> percentile line on Figures 8 and 9, but not for the 25<sup>th</sup> percentile line. The effect of the restriction is not apparent for the 10-year nominal interest rates. The level of uncertainty increases over the 50 year time period used in the forecast. Since the nominal interest rate is determined by adding the real interest rate to the inflation rate, the increasing uncertainty reflected by real interest rates and the inflation rate generates the same behavior for nominal interest rates.

The histograms for the 3-month, 1-year, and 10-year model nominal interest rates and the actual 3-month, 1-year, and 10-year nominal interest rates are displayed in Figures 11-13. (The 1-month values are not consistently available for historical data over a long enough time period

to be relevant. Therefore, 3-month interest rates are used for in Figure 11.) Figures 11-13 show the distribution of nominal interest rates one year into the projection period.

Significant differences do exist between the modeled and historical distributions for interest rates. In Figure 11, the modeled 3-month nominal interest rates are 0 in almost 20% of the cases, whereas actual 3-month interest rates have never been below 0.5 percent (the column reflecting the 1% bin represents values between 0.5 and 1.5 percent). However, combining the model values for 0 and 1 percent indicates a total in line with actual values. In addition, the model distributions are smoother than the actual values, which is natural since the model results are based on 5,000 iterations whereas the actual results, even though derived from 845 (monthly) or 614 (1 and 10 year) observations, are not at nearly as smooth, indicating that the system that generates interest rates is not as straightforward as the model.

At first glance, modeled interest rates are generally lower than the historical rates. It is important to note that the modeled interest rates are influenced by the starting values for the initial real interest rate ( $r_{init1}$ ), the initial mean reversion level for the real interest rate ( $r_{init2}$ ), and the initial inflation level ( $q_{init1}$ ), which are lower than historical averages.

The comparison between the 10-year modeled rates and the 10-year historical rates, Figure 13, indicates a few differences. The modeled interest rates are more compact than actual 10-year interest rates have been. If the user feels that the variance of the model values should be closer to the historical distribution, then the strength of the mean reversion factor in the interest rate model can be reduced, but this would increase the incidence of negative interest rates unless the user selects to avoid negative nominal interest rates. The other significant difference is the skewness. The historical rates exhibit positive skewness, but the modeled rates have a slight negative skewness. Finally, the model rates are lower than historical values, again due to starting with the current low levels of interest rates.

### **Stock Returns and Dividends**

The values for large and small stock returns indicate, as expected, higher average returns and greater variability for small stocks. As shown on Table 1, the mean of the initial values (after one year) of large stocks is 8.7% and of small stocks is 13.4%. The mean of the large stock values increases to 11.6% at the end of 50 years and for small stocks increases to 13.6%.

The 1<sup>st</sup> – 99<sup>th</sup> percentile range after 10 years is –15.9% to 29.6% for large stocks and –15.9% to 39.7% for small stocks.

The Funnel of Doubt graphs, Figures 14 and 15, indicate an inverted funnel, compared to the displays of interest rates and inflation. This means that uncertainty reduces over time and is due to the way the values are calculated. The projected values shown are *geometric average* returns for large and small stocks over the projection period. For example, the 1-year values are returns over a one year period, the 10-year values are average annual returns over the ten year period, etc. Thus, Figures 14 and 15 show that the average annual returns expected over a 50 year period are much more predictable than those for a 1-year period.

Histograms of the 1-year returns for the large (Figure 16) and small (Figure 17) stock returns as generated by the model are displayed, along with actual 1-year returns for 500 large stocks for 1871-2004 and small stock returns over the period 1926-2003. The large stocks are based on the S&P 500 (or a sample chosen to behave similarly for the years prior to the construction of the S&P 500). The data are available online at a website generated by Robert Shiller, author of *Irrational Exuberance* ([http://www.econ.yale.edu/~shiller/data/ie\\_data.htm](http://www.econ.yale.edu/~shiller/data/ie_data.htm)). The small stock values are based on Ibbotson's *Stocks, Bonds and Bills*. The graphs for large stocks (Figure 16) are relatively similar, although, as would be expected, the results of the 5,000 iterations of the model produce a smoother distribution. The histograms for small stocks (Figure 17) show that historical values have been more variable, with a notable outlier at 190% return, which represents a single observation. The model values also have single observations around that level, but no one bin produces as large a proportion of the outcomes as the one occurrence out of 78 years of the historical experience to be as obvious on the graph.

The dividend yield for equities is 1.5% for the first year and 2.3% for the last year values. The 1<sup>st</sup> – 99<sup>th</sup> percentile range after 10 years is 0.6% to 3.9%. The Funnel of Doubt graph of the dividend yield, Figure 18, increases over time as interest rates and inflation do. Figure 19 displays the histogram of the modeled dividend yields and the actual dividend yields over the period 1871-2003, based on data available from Robert Shiller. Historically, dividend yields have varied more widely than the model predicts and have been centered at a higher level. This may be a result, in part, of structural shift in the dividend payment history in the US. Bernstein (1996) notes that prior to the late 1950s, dividends tended to be higher than interest rates on corporate bonds. This was based on the understanding that stocks were riskier than bonds and

therefore should pay a higher return. Since 1959, though, dividend yields have tended to be lower than interest rates, ranging from 1.1% to 5.4%, which is in line with the simulation results.

### **Unemployment and Real Estate Returns**

The mean value of the unemployment rate, as shown on Table 1, begins at 6.0% and increases to 6.1% (which is the long-run mean value) for the end of 50 years. The 1<sup>st</sup> – 99<sup>th</sup> percentile range after 10 years is 3.5% to 8.7%. Figure 20 shows the Funnel of Doubt graph, neither increases over time (as interest rates and inflation do) nor decreases (as stock returns do). The histogram of modeled unemployment rates, along with the distribution of actual values over the period 1948-2003 are shown in Figure 21. The historical values represent the unemployment rate each January from 1948-2004. By selecting only a single unemployment rate from each year, the frequency of the historical values corresponds with that of the model values, which are the unemployment rates indicated after the first year of the model run. Although the actual unemployment rates have varied a bit more than the model results do, the distributions are quite similar.

Real estate returns are the final variable included in the financial scenario model. From Table 1, the mean value of real estate returns is 8.1% in the first year and 9.4% after 50 years. The 1<sup>st</sup> – 99<sup>th</sup> percentile range after 10 years is 3.0% to 16.1%. The Funnel of Doubt graph, Figure 22, is similar to the returns on stocks, for the same reasons. The histograms of modeled results and the actual returns based on the National Index from National Council of Real Estate Investment Fiduciaries (NCREIF) for 1978-2003 are shown on Figure 23. The model values show a smooth distribution that is centered about the historical returns. Unfortunately, only 26 years of annual returns are available, so it is difficult to draw any conclusions on the fit.

### **Correlations**

Table 2 displays the correlation matrix for all the output variables at the end of the first projection year (row 16 of the spreadsheet). Table 3 displays the corresponding matrix from history over the period April 1953 – December 2001. Stock data is based on Ibbotson and interest rates and inflation are from St. Louis Federal Reserve Data. Several conclusions can be drawn about the validity of the model based on a comparison of the two correlation matrices.

First, the historical correlation between large and small stocks is .744. The correlation between the model values of large and small stocks is .698, which looks quite reasonable.

The correlation between inflation and T-bills has been .593 historically. This correlation is also clearly reflected in the model values, with a correlation of .906 between the one month inflation rate and the 1-month nominal interest rate, .892 between the 1-year inflation rate and the 1-year nominal interest rate, and .617 between the 10-year inflation rate and the 10-year nominal interest rate. Since the nominal interest is the sum of the real interest rate and the inflation rate, and the real interest rate is constrained to be no less than the negative of the inflation rate, this correlation is built into the model.

Historically, T-bill rates and stock returns have been negatively correlated (-0.078 for large stocks and -0.065 for small stocks). In the model, there was a slight positive correlation between the 1-year nominal interest rate and stock returns (0.099 for large stocks and 0.087 for small stocks). Also, the historical correlation between inflation and stock returns has been negative (-0.138 for large stocks and -0.100 for small stocks). The correlation in the model values between the 1-year inflation rate and large stocks was 0.089 and 0.076 for small stocks.

### **Alternate Parameters**

The base parameters provide one feasible set of values to use in modeling future economic conditions. These should be viewed as a starting point in these applications. However, users should develop an understanding of the impact of the different parameters and then adjust these parameters as necessary to generate distributions that are suitable for the particular applications of the model.

## **6. CONCLUSION**

Historically, actuaries tended to use deterministic calculations to value financial products. As technology improved, actuaries began to incorporate different assumptions about insurance and economic variables that would lead to several distinct scenarios to better measure financial risk. The explosion of computing power now gives actuaries and other financial analysts tremendous tools for more refined risk analyses. Modern approaches to financial modeling begin by specifying the underlying economic and financial environments based on sophisticated mathematical equations, and then incorporate product-specific features which are commonly

related to those external conditions. This approach yields a much richer understanding of the risks associated with financial products.

The financial scenario model, and its underlying mathematical structure, presented in this paper provide an integrated framework for sampling from a wide range of future financial scenarios. The model produces output values for interest rates, inflation, stock and real estate returns, dividends, and unemployment. The model can be incorporated into a variety of insurance applications, including dynamic financial analysis, cash flow testing, solvency testing, and operational planning. It is hoped that this work will facilitate the use of recent advances in economic and financial modeling into the actuarial profession.

## REFERENCES

- Ahlgrim, Kevin, and Stephen D'Arcy, 2003, "Pension obligations squeeze state budget," *Illinois Tax Facts* (October). <http://www.taxpayfedil.org/oct03.htm>
- Ahlgrim, Kevin, Stephen D'Arcy, and Richard Gorvett, 1999, "Parameterizing Interest Rate Models," *Casualty Actuarial Society Forum*, Summer 1-50. <http://www.casact.org/pubs/forum/99sforum/99sf001.pdf>
- Alexander, Carol, 2001, *Market Models: A Guide to Financial Data Analysis*, Chichester, England: John Wiley & Sons.
- Ait-Sahalia, Yacine, 1996, "Testing continuous-time models of the spot interest rate," *Review of Financial Studies*, 9: 385-426.
- Ait-Sahalia, Yacine, 1999, "Do Interest Rates Really Follow Continuous-Time Markov Diffusions?" University of Chicago Working Paper
- Bernstein, Peter, 1996, *Against the Gods: The Remarkable Story of Risk*, New York: John Wiley & Sons, Inc.
- Black, Fischer, and Myron Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81: 637-654.
- Campbell, John, Andrew Lo, and A. Craig MacKinlay, 1997, *The Econometrics of Financial Markets*, Princeton, NJ: Princeton University Press.
- Casualty Actuarial Society Financial Analysis Committee (CASFAC), 1989, "A Study of the Effects of Asset/Liability Mismatch on Property/Casualty Insurance Companies," *Valuation Issues*, 1-52. <http://www.casact.org/pubs/dpp/dpp89/89dpp001.pdf>
- Chan, K. C., G. Andrew Karolyi, Francis Longstaff, and Anthony Sanders, 1992, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate," *Journal of Finance*, 47: 1209-1227.
- Chapman, David, and Neil Pearson, 2000, "Is the short rate drift actually nonlinear?" *Journal of Finance*, 55: 355-388.
- Chapman, David, and Neil Pearson, 2001, "What Can Be Learned from Recent Advances in Estimating Models of the Term Structure," *Financial Analysts Journal* (July/August), 77-95.
- Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53: 385-407.

D'Arcy, Stephen, Richard Gorvett, Thomas Hettinger, and Robert Walling III, 1998, "Using the Public Access Dynamic Financial Analysis Model: A Case Study," *CAS Dynamic Financial Analysis Call Paper Program*, pp. 53-118.

D'Arcy, Stephen, Richard Gorvett, Joseph Herbers, Thomas Hettinger, Steven Lehmann, and Michael Miller, 1997, "Building a Public Access PC-Based DFA Model," *CAS Dynamic Financial Analysis Task Special Interest Seminar*, pp. 1-40.

Fama, Eugene, 1984, "The Information in the Term Structure," *Journal of Financial Economics* 13: 509-528.

Fama, Eugene, 1990, "Term Structure Forecasts of Interest Rates, Inflation, and Real Returns," *Journal of Monetary Economics* 25: 59-76.

Fama, Eugene, and Robert Bliss, 1987, "The Information in Long-Maturity Forward Rates," *American Economic Review* 77: 680-692.

Fisher, Irving, 1930, *Theory of Interest*, New York: The Macmillan Company.

Fitton and McNatt, 1997, "The Four Faces of an Interest Rate Model," Chapter 2 in *Advances in Fixed Income Valuation: Modeling and Risk Management*, F. Fabozzi, ed, Frank J. Fabozzi Assoc.

Hardy, Mary, 2001, "A Regime-Switching Model of Long-Term Stock Returns," *North American Actuarial Journal*, 5(2): 41-53. [http://www.soa.org/library/naaj/1997-09/naaj0104\\_4.pdf](http://www.soa.org/library/naaj/1997-09/naaj0104_4.pdf)

Heath, David, Robert Jarrow, and Andrew Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60: 77-105.

Hibbert, John, Philip Mowbray, and Craig Turnbull, 2001, "A Stochastic Asset Model & Calibration for Long-Term Financial Planning Purposes," Technical Report, Barrie & Hibbert Limited.

Ho, Thomas, and Sang-Bin Lee, 1986, "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41: 1011-1029.

Hodes, Douglas, and Sholom Feldblum, 1996, "Interest Rate Risk and Capital Requirements for Property/Casualty Insurance Companies," *Proceedings of the Casualty Actuarial Society*, 83:490-562. <http://www.casact.org/pubs/proceed/proceed96/96490.pdf>

Hull, John, 2003, *Options, Futures, and Other Derivatives*, 5<sup>th</sup> edition, Upper Saddle River, NJ: Prentice Hall.

Hull, John, and Alan White, 1990, "Pricing Interest-Rate-Derivative Securities," *Review of Financial Studies*, 3: 573-592.

Hull, John, and Alan White, 1994, "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives* (Winter), 37-48.

Jegadeesh, Narasimhan, 1998, "An Empirical Analysis of the Pricing of Interest Rate Caps," University of Illinois at Urbana-Champaign Working Paper.

Litterman, Robert, and Jose Scheinkman, 1991, "Common Factors Affecting Bond Returns," *Journal of Fixed Income* (June), 54-61.

Pennacchi, George, 1991, "Identifying the Dynamics of Real Interest Rates and Inflation: Evidence Using Survey Data," *Review of Financial Studies*, 4: 53-86.

Redington, Frank, 1952, "Review of the principles of life office valuations," *Journal of the Institute of Actuaries*, 78: 1-40.

Reitano, Robert, 1992, "Non-parallel Yield Curve Shifts and Immunization," *Journal of Portfolio Management*, 18: 36-43.

Risa, Stefano, 2001, "Nominal and Inflation Indexed Yields: Separating Expected Inflation and Inflation Risk Premia," Columbia University Working Paper.

Santomero, Anthony, and David Babbel, 1997, "Financial Risk Management: An Analysis of the Process," *Journal of Risk and Insurance*, 64:231-270.

Siegel, Jeremy, 2002, *Stocks for the Long Run*, New York: McGraw-Hill.

Shiller, Robert, 2000, *Irrational Exuberance*, Princeton, NJ: Princeton University Press.

Sornette, Didier, 2003, *Why Stock Markets Crash: Critical Events in Complex Financial Systems*, Princeton, NJ: Princeton University Press.

Staking, Kim, and David Babbel, 1995, "The Relation Between Capital Structure, Interest Rate Sensitivity, and Market Value in the Property-Liability Insurance Industry," *Journal of Risk and Insurance*, 62:690-718.

Stanton, Richard, 1997, "A nonparametric model of term structure dynamics and the market price of interest rate risk," *Journal of Finance*, 52: 1973-2002.

Tilley, James, 1988, "The Application of Modern Techniques to the Investment of Insurance and Pension Funds" (New York: Morgan Stanley & Co.)

Tuckman, Bruce, 1996, *Fixed Income Securities*, New York: John Wiley & Sons, Inc.

Vasicek, Oldrich, 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5: 177-188.

Wilkie, David, 1986, "A Stochastic Investment Model for Actuarial Use," *Transactions of the Faculty of Actuaries*, 39: 341-403.

Wilkie, David, 1995, "More on a Stochastic Model for Actuarial Use," *British Actuarial Journal*, pp. 777-964

Wilmott, Paul, 1998, *Derivatives: The Theory and Practice of Financial Engineering*, Chichester, England: John Wiley & Sons.

Yan, Hong, 2001, "Dynamic Models of the Term Structure," *Financial Analysts Journal*, July/August: 60-74.